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Simple Continued Fractions for Some Irrational Numbers, II

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In this paper we prove a theorem allowing us to determine the continued fraction expansion for $\sum_{k=0}^{\infty} u^{-c(k)}$, where $c(k)$ is any sequence of positive integers that grows sufficiently quickly. As an application, we determine the continued fraction expansion for Liouville's famous transcendental number $\sum_{k=0}^{\infty} m^{-(k+1)!}$.

I. INTRODUCTION

In a previous paper [1], the author discussed the continued fractions for the irrational numbers defined by

$$B(u, \infty) = \sum_{k=0}^{\infty} u^{-2^k} \qquad (u \geqslant 2, \text{ an integer}).$$

In this paper, we prove a more general result concerning the continued fractions for the irrational numbers

$$S(u, \infty) = \sum_{k=0}^{\infty} u^{-c(k)},$$

where $c(k)$ is any sequence of positive integers that grows sufficiently quickly. As an application, we determine the continued fraction for Liouville's number.

II. A THEOREM

Let $\{c(k)\}_{k=0}^{\infty}$ be a sequence of positive integers such that $c(v+1) \geqslant 2c(v)$ for all $v \geqslant v'$, where v' is a non-negative integer. Let $d(v) = c(v+1) - 2c(v)$. Define $S(u, v)$ as follows:

$$S(u, v) = \sum_{k=0}^{v} u^{-c(k)}.$$

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Then we have the following:

THEOREM. *Suppose $v \geqslant v'$. If $S(u, v) = [a_0, a_1, a_2, \ldots, a_n]$ and n is even, then*

$$S(u, v+1) = [a_0, a_1, \ldots, a_n, u^{d(v)} - 1, 1, a_n - 1, a_{n-1}, a_{n-2}, \ldots, a_2, a_1].$$

Proof. This proof uses the ideas behind the proof of Theorem 1 in [1]. Let $c(v+1) > 2c(v)$, and let

$$S(u, v) = [a_0, a_1, \ldots, a_n] = p_n/q_n.$$

We find

$$[a_0, a_1, \ldots, a_n, u^{d(v)} - 1] = \frac{(u^{d(v)} - 1)\, p_n + p_{n-1}}{(u^{d(v)} - 1)\, q_n + q_{n-1}},$$

$$[a_0, a_1, \ldots, a_n, u^{d(v)} - 1, 1] = \frac{u^{d(v)} p_n + p_{n-1}}{u^{d(v)} q_n + q_{n-1}},$$

$$\begin{aligned}
&[a_0, a_1, \ldots, a_n, u^{d(v)} - 1, 1, a_n - 1] \\
&\quad = \frac{(a_n - 1)(u^{d(v)} p_n + p_{n-1}) + (u^{d(v)} - 1)\, p_n + p_{n-1}}{(a_n - 1)(u^{d(v)} q_n + q_{n-1}) + (u^{d(v)} - 1)\, q_n + q_{n-1}} \\
&\quad = \frac{u^{d(v)} a_n p_n + a_n p_{n-1} - p_n}{u^{d(v)} a_n q_n + a_n q_{n-1} - q_n},
\end{aligned}$$

$$\begin{aligned}
&[a_0, a_1, \ldots, a_n, u^{d(v)} - 1, 1, a_n - 1, a_{n-1}, a_{n-2}, \ldots, a_2, a_1] \\
&\quad = \frac{(u^{d(v)} p_n + p_{n-1})\, q_{n-2} + (u^{d(v)} a_n p_n + a_n p_{n-1} - p_n)\, q_{n-1}}{(u^{d(v)} q_n + q_{n-1})\, q_{n-2} + (u^{d(v)} a_n q_n + a_n q_{n-1} - q_n)\, q_{n-1}} \\
&\quad = \frac{u^{d(v)} p_n q_n + 1}{u^{d(v)} q_n^2}.
\end{aligned}$$

In this last simplification, we use the fact that

$$p_{n-1} q_{n-2} - p_{n-2} q_{n-1} = 1$$

if n is even, and Eqs. (4)–(7) of [1].

Now

$$\begin{aligned}
p_n/q_n &= \sum_{k=0}^{v} u^{-c(k)} \\
&= u^{-c(v)} \left(1 + u \sum_{k=0}^{v-1} u^{c(v) - c(k) - 1}\right)
\end{aligned}$$

and $(p_n, q_n) = 1$; hence $q_n = u^{c(v)}$. Therefore,

$$\begin{aligned}[][a_0, a_1, \ldots, a_n, u^{d(v)} - 1, 1, a_n - 1, a_{n-1}, a_{n-2}, \ldots, a_2, a_1] \\ &= \frac{p_n}{q_n} + \frac{1}{u^{d(v)} q_n^2} \\ &= S(u, v) + u^{-c(v+1)} \\ &= S(u, v+1).\end{aligned}$$

It should be noted that zero elements appear in the expansion of $S(u, v+1)$ if $d(v) = 0$. These zero elements may be removed with the use of the formula

$$[a_0, a_1, \ldots, a_k, 0, a_{k+1}, \ldots] = [a_0, a_1, \ldots, a_k + a_{k+1}, a_{k+2}, \ldots].$$

Thus, the theorem holds for $c(v) = 2^v$ (as shown in [1]). This proves the theorem for $c(v+1) \geqslant 2c(v)$.

One hypothesis of the theorem states that n is even. This is not really much of a restriction, since any finite continued fraction can be expressed in two equivalent forms:

If $a_n \geqslant 2$,

$$[a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_n - 1, 1].$$

If $a_n = 1$,

$$[a_0, a_1, \ldots, a_{n-1}, 1] = [a_0, a_1, \ldots, a_{n-2}, a_{n-1} + 1],$$

for example, see Hardy and Wright [2]. Thus we can always write a continued fraction for $S(u, v)$ such that n is even.

Repeated application of the theorem gives longer and longer prefixes of the continued fraction for $S(u, \infty)$, since it is easily seen that the expansions for $S(u, v)$ and $S(u, \infty)$ coincide up to the penultimate partial denominator in the expansion for $S(u, v)$.

III. An Application

Using the theorem, we calculate the continued fraction expansion for the Liouville transcendental number [3]

$$L(m) = \sum_{k=0}^{\infty} m^{-(k+1)!} \qquad (m \text{ an integer } \geqslant 2).$$

Let $c(k) = (k+1)!$. Then $d(k) = k(k+1)!$. Repeated application of the theorem gives

$$S(u,0) = m^{-1!} = [0, m] = [0, m-1, 1],$$

$$S(u,1) = m^{-1!} + m^{-2!} = [0, m-1, 1, 0, 1, 0, m-1]$$
$$= [0, m-1, m+1],$$

$$S(u,2) = m^{-1!} + m^{-2!} + m^{-3!}$$
$$= [0, m-1, m+1, m^2-1, 1, m, m-1].$$

We may continue in this fashion. Hence

$$L(m) = [0, m-1, m+1, m^2-1, 1, m, m-1, m^{12}-1, 1, m-2, \ldots].$$

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